Delbrück scattering in a strong external field

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(Received 7 October 1991)

We evaluate the Delbrück scattering amplitude to all orders of the interaction with the external field of a nucleus employing nonperturbative electron Green's functions. The results are given analytically in form of a multipole expansion.

PACS number(s): 12.20.Ds; 13.60.Fz

I. INTRODUCTION

Delbrück scattering [1], the scattering of a photon by the static electromagnetic field of a nucleus, is one of the fundamental processes of quantum electrodynamics (QED) which remains to be calculated. A few theoretical results exist, involving various approximations, e.g., forward scattering, small- or large-angle scattering, high-energy scattering, as well as the Born approximation, which is most frequently used and which neglects multiple interactions with the external field. For a review of those results we refer, e.g., to Papatzacos and Mork [2]. Cheng *et al.* [3] calculated Delbrück scattering in the Born approximation without any additional assumptions.

In 1957 Rohrlich [4] estimated the "Coulomb corrections" to the lowest-order Delbrück scattering amplitude for forward scattering of a photon on a lead nucleus. Within the last two decades several other attempts have been pursued to go beyond the Born approximation. Cheng and Wu [5–7] derived an expression for the Delbrück scattering amplitude valid at high energies and small momentum transfers, which is exact to all orders in $(Z\alpha)$, viz., including multiphoton exchange to arbitrary orders. Cheng *et al.* [3] and Milstein and co-workers [8,9] investigated the high-energy process at large scattering angles.

Delbrück scattering is one of the few nonlinear processes of QED which is observable. Experimental results [10, 11] suggest that the Born approximation is insufficient to describe the data in particular for high-Z nuclei; thus, multiphoton exchange has to be taken into account.

In our approach we employ the nonperturbative formalism of Wichmann and Kroll [12] to evaluate the amplitude for Delbrück scattering. Feynman diagrams of all orders in the coupling constant $(Z\alpha)$ to the external field of the nucleus are included, a method that has proved to be rather successful for the theoretical determination of self-energy [13–16] and vacuum-polarization corrections [12, 17, 18] in high-Z atoms. This method provides a correct description of Delbrück scattering for the entire range of Z. Furthermore Green's functions for the field of an extended nucleus are easily incorporated to include finite nuclear-size effects.

II. THEORY OF DELBRÜCK SCATTERING

A. Delbrück scattering-matrix element

Several processes contribute to the scattering of photons on atoms: Delbrück scattering, Compton scattering off the nucleus (Thomson scattering), Rayleigh scattering off the bound electrons, and excitation of nuclear resonances. Since all these processes add up coherently, it is necessary to know explicitly the real and imaginary part of the corresponding matrix elements. The Feynman diagram of Delbrück scattering is shown in Fig. 1. The corresponding S-matrix element is $(\hbar = c = 1)$

$$S_{\text{DBS}} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d^4x_1 d^4x_2 \,\mathrm{Tr}\,[ie\gamma_\nu\, S_F(x_1,x_2)\, ie\gamma_\mu\, S_F(x_2,x_1)]\, \varepsilon^\nu_{\lambda_1}(\mathbf{k}_1) e^{-ik_1x_1}\, [\varepsilon^\mu_{\lambda_2}(\mathbf{k}_2)]^* e^{ik_2x_2}, \tag{1}$$

where k_1 and k_2 are the four-momenta of the incoming and outgoing photon, respectively, λ_1 and λ_2 are their polarizations, and $\varepsilon^\mu_{\lambda_i}$ are the photon polarization four-vectors. $S_F(x_1,x_2)$ designates the Feynman propagator of the electron-positron field in the external field of the nucleus (see Appendix A) and e is the electron charge. Making use of (A1) we obtain

$$S_{\text{DBS}} = 4\pi\alpha\,\delta(\omega_2-\omega_1)\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d^3x_1 d^3x_2 \int_{-\infty}^{+\infty} dz\,\varepsilon^{\nu}_{\lambda_1}(\mathbf{k}_1)e^{i\mathbf{k}_1\cdot\mathbf{x}_1}[\varepsilon^{\mu}_{\lambda_2}(\mathbf{k}_2)]^* e^{-i\mathbf{k}_2\cdot\mathbf{x}_2}$$
$$\times\text{Tr}\left[\mathcal{G}(\mathbf{x}_1,\mathbf{x}_2,z\pm i\varepsilon)i\gamma_\mu\gamma_0\,\mathcal{G}(\mathbf{x}_2,\mathbf{x}_1,z+\omega\pm i\varepsilon)i\gamma_\nu\gamma_0\right], \quad (2)$$

where ω_1 and ω_2 are the energies of the photons, $\mathcal{G}(\mathbf{x}_1,\mathbf{x}_2,z)$ is the electron Green's function defined in Appendix A and $\alpha = e^2/4\pi$ is the fine-structure constant. Since we are interested in the scattering of photons by a static field, we transform to the T matrix element for elastic scattering:

$$S_{\text{DBS}} = 1 + i\delta(\omega_2-\omega_1)T \quad . \quad (3)$$

Now the differential cross section for Delbrück scattering becomes

$$\left(\frac{d\sigma}{d\Omega}\right)_{\text{DBS}} = \frac{1}{64\pi^4}|T|^2 = \left(\frac{\alpha}{2\pi}\right)^2|M|^2, \quad (4)$$

where we have defined a scaled matrix element M:

$$T = 4\pi\alpha M \quad . \quad (5)$$

Delbrück scattering is described completely by the matrix elements for circular polarized photons. Since the $\varepsilon^0_{\lambda_i}$ component of the polarization four-vector vanishes for right- and left-handed polarizations $\lambda_i = \pm 1$, we obtain, for M,

$$M = i\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} d^3x_1 d^3x_2 \int_{-\infty}^{+\infty} dz\, e^{i\mathbf{k}_1\cdot\mathbf{x}_1}\, e^{-i\mathbf{k}_2\cdot\mathbf{x}_2}\text{Tr}\{\,\mathcal{G}(\mathbf{x}_1,\mathbf{x}_2,z\pm i\varepsilon)\boldsymbol{\alpha}\cdot[\boldsymbol{\varepsilon}_{\lambda_2}(\mathbf{k}_2)]^*$$
$$\times\mathcal{G}(\mathbf{x}_2,\mathbf{x}_1,z+\omega\pm i\varepsilon)\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}_{\lambda_1}(\mathbf{k}_1)\}, \quad (6)$$

where ω is the energy of the photons. Employing the partial-wave decomposition of the Green's function (A2) we find, for the components of the trace,

$$\text{Tr}[\mathcal{G}(\mathbf{x}_1,\mathbf{x}_2,z\pm i\varepsilon)\alpha^m\,\mathcal{G}(\mathbf{x}_2,\mathbf{x}_1,z+\omega\pm i\varepsilon)\alpha^n] = \sum_{\kappa_1=-\infty}^{+\infty}\sum_{\kappa_2=-\infty}^{+\infty}\sum_{i=1}^{2}\sum_{j=1}^{2}\text{Tr}[B^{\kappa_1\kappa_2}_{ij}(r_1,r_2,z)\,{}^{nm}C^{\kappa_1\kappa_2}_{ij}(\Omega_1,\Omega_2)] \quad (7)$$

with the abbreviations

$$\begin{aligned}
B^{\kappa_1\kappa_2}_{11}(r_1,r_2,z) &= \mathcal{G}^{12}_{\kappa_1}(r_2,r_1,z+\omega)\mathcal{G}^{12}_{\kappa_2}(r_1,r_2,z),\\
B^{\kappa_1\kappa_2}_{12}(r_1,r_2,z) &= \mathcal{G}^{11}_{\kappa_1}(r_2,r_1,z+\omega)\mathcal{G}^{22}_{\kappa_2}(r_1,r_2,z),\\
B^{\kappa_1\kappa_2}_{21}(r_1,r_2,z) &= \mathcal{G}^{22}_{\kappa_1}(r_2,r_1,z+\omega)\mathcal{G}^{11}_{\kappa_2}(r_1,r_2,z),\\
B^{\kappa_1\kappa_2}_{22}(r_1,r_2,z) &= \mathcal{G}^{21}_{\kappa_1}(r_2,r_1,z+\omega)\mathcal{G}^{21}_{\kappa_2}(r_1,r_2,z)
\end{aligned} \quad (8)$$

and

$$\begin{aligned}
{}^{nm}C^{\kappa_1\kappa_2}_{11}(\Omega_1,\Omega_2) &= -\pi_{\kappa_1,-\kappa_1}(2,1)\sigma_n\pi_{\kappa_2,-\kappa_2}(1,2)\sigma_m,\\
{}^{nm}C^{\kappa_1\kappa_2}_{12}(\Omega_1,\Omega_2) &= \pi_{\kappa_1,\kappa_1}(2,1)\sigma_n\pi_{-\kappa_2,-\kappa_2}(1,2)\sigma_m,\\
{}^{nm}C^{\kappa_1\kappa_2}_{21}(\Omega_1,\Omega_2) &= \pi_{-\kappa_1,-\kappa_1}(2,1)\sigma_n\pi_{\kappa_2,\kappa_2}(1,2)\sigma_m,\\
{}^{nm}C^{\kappa_1\kappa_2}_{22}(\Omega_1,\Omega_2) &= -\pi_{-\kappa_1,\kappa_1}(2,1)\sigma_n\pi_{-\kappa_2,\kappa_2}(1,2)\sigma_m,
\end{aligned} \quad (9)$$

where r_1 and r_2 are the radial variables of the incoming and outgoing photon, and Ω_1 and Ω_2 are the corresponding solid angles. The $\mathcal{G}^{ij}_{\kappa}(r_1,r_2,z)$ denote the components of the radial Green's functions; the functions $\pi_{\kappa_i,\kappa_j}(1,2)$ are defined in (A3), and the σ_i are the Pauli matrices.

We use the decomposition of the circularly polarized plane wave propagating in direction $\mathbf{k}$ into angular-momentum eigenfunctions [19]:

$$e^{i\mathbf{k}\cdot\mathbf{x}}\boldsymbol{\varepsilon}_\lambda(\mathbf{k}) = \sum_{j=1}^{\infty}\sum_{m=-j}^{+j}\sum_{l=j-1}^{j+1} i^{j-1}\sqrt{2\pi(2j+1)}D^j_{m\lambda}(\hat{k})$$
$$\times t^\lambda_{j,l}\,\mathbf{Y}^{ljm}(\Omega)\,j_l(\omega|\mathbf{x}|) \quad (10)$$

with

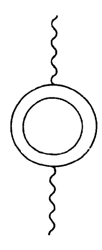

FIG. 1. Feynman diagram of Delbrück scattering. The wavy line denotes a photon; the double line denotes an electron in the external field of a nucleus.

$$t^{\lambda}_{j,j-1} = \sqrt{\frac{j+1}{2j+1}},$$
$$t^{\lambda}_{j,j} = -i\lambda, \tag{11}$$
$$t^{\lambda}_{j,j+1} = -\sqrt{\frac{j}{2j+1}},$$

where $D^{j}_{m_1 m_2}(\hat{k})$ are the Wigner rotation matrices, $\mathbf{Y}^{ljm}(\Omega)$ are the vector spherical harmonics, and $j_l(x)$ are spherical Bessel functions. We shall make the abbreviation

$$T^{\lambda}_{jlm} = i^{j-1}\sqrt{2\pi(2j+1)}D^{j}_{m\lambda}(\hat{k})\,t^{\lambda}_{j,l} \quad . \tag{12}$$

Now we can isolate the part of the Delbrück scattering amplitude depending on the angular variables of the photons

$$D^{\kappa_1\kappa_2}_{ij} = \sum_{n=1}^{3}\sum_{m=1}^{3}\int_{4\pi}\int_{4\pi} d\Omega_1 d\Omega_2 \mathrm{Tr}\left[{}^{nm}C^{\kappa_1\kappa_2}_{ij}(\Omega_1,\Omega_2)\right] \times \mathbf{Y}^{l_1 j_1 m_1}_{n}(\Omega_1)[\mathbf{Y}^{l_2 j_2 m_2}_{m}(\Omega_2)]^{*}. \tag{13}$$

With these angular coefficients, M can be expressed by

$$M = i\int_0^{\infty}\int_0^{\infty} dr_1 dr_2 \int_{-\infty}^{+\infty} dz\, r_1^2 r_2^2 \sum_{j_1=1}^{\infty}\sum_{m_1=-j_1}^{+j_1}\sum_{l_1=j_1-1}^{j_1+1}\sum_{j_2=1}^{\infty}\sum_{m_2=-j_2}^{+j_2}\sum_{l_2=j_2-1}^{j_2+1} T^{\lambda_1}_{j_1 l_1 m_1}\left(T^{\lambda_2}_{j_2 l_2 m_2}\right)^{*} j_{l_1}(\omega r_1) j_{l_2}(\omega r_2) \times \sum_{\kappa_1=-\infty}^{+\infty}\sum_{\kappa_2=-\infty}^{+\infty}\sum_{i=1}^{2}\sum_{j=1}^{2} B^{\kappa_1\kappa_2}_{ij}(r_1,r_2,z)D^{\kappa_1\kappa_2}_{ij}. \tag{14}$$

Performing the integration over the solid angles of the photons and writing the $D^{\kappa_1\kappa_2}_{ij}$ in terms of new functions $D^{\kappa_1\kappa_2}_{\sigma}$ with $\sigma = \pm 1$,

$$D^{\kappa_1\kappa_2}_{11} = -D^{-\kappa_1-\kappa_2}_{-1}, \quad D^{\kappa_1\kappa_2}_{12} = D^{\kappa_1-\kappa_2}_{+1},$$
$$D^{\kappa_1\kappa_2}_{21} = D^{-\kappa_1\kappa_2}_{+1}, \quad D^{\kappa_1\kappa_2}_{22} = -D^{\kappa_1\kappa_2}_{-1}, \tag{15}$$

yields

$$D^{\kappa_1\kappa_2}_{\sigma} = 6\,\delta_{jj_1}\delta_{j_1j_2}\,\delta_{mm_1}\delta_{m_1m_2}\,[2j(\kappa_1)+1][2j(\kappa_2)+1]\langle l(\kappa_1)\|Y^{l_1}\|l(\sigma\kappa_2)\rangle\langle l(\sigma\kappa_1)\|Y^{l_2}\|l(\kappa_2)\rangle \times \begin{Bmatrix} l(\kappa_1) & l(\sigma\kappa_2) & l_1 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ j(\kappa_1) & j(\kappa_2) & j \end{Bmatrix}\begin{Bmatrix} l(\sigma\kappa_1) & l(\kappa_2) & l_2 \\ \frac{1}{2} & \frac{1}{2} & 1 \\ j(\kappa_1) & j(\kappa_2) & j \end{Bmatrix}, \tag{16}$$

where $l(\kappa)$ and $j(\kappa)$ are the orbital and total angular-momentum being related to the Dirac angular momentum quantum number κ, respectively. The reduced matrix element of the spherical harmonics with angular momentum k between states of angular momentum k_1 and k_2 is denoted by $\langle k_1\|Y^{k}\|k_2\rangle$ and the curly brackets are $9j$ symbols. The matrix element now reads

$$M = i\int_0^{\infty}\int_0^{\infty} dr_1 dr_2 \int_{-\infty}^{+\infty} dz\, r_1^2 r_2^2 \sum_{j_1=1}^{\infty}\sum_{m_1=-j_1}^{+j_1}\sum_{l_1=j_1-1}^{j_1+1}\sum_{j_2=1}^{\infty}\sum_{m_2=-j_2}^{+j_2}\sum_{l_2=j_2-1}^{j_2+1} T^{\lambda_1}_{j_1 l_1 m_1}\left(T^{\lambda_2}_{j_2 l_2 m_2}\right)^{*} j_{l_1}(\omega r_1) j_{l_2}(\omega r_2) \times \sum_{\kappa_1=-\infty}^{+\infty}\sum_{\kappa_2=-\infty}^{+\infty}[-B^{\kappa_1\kappa_2}_{11}(r_1,r_2,z)D^{-\kappa_1-\kappa_2}_{-1} + B^{\kappa_1\kappa_2}_{12}(r_1,r_2,z)D^{\kappa_1-\kappa_2}_{+1} + B^{\kappa_1\kappa_2}_{21}(r_1,r_2,z)D^{-\kappa_1\kappa_2}_{+1} - B^{\kappa_1\kappa_2}_{22}(r_1,r_2,z)D^{\kappa_1\kappa_2}_{-1}]. \tag{17}$$

We define the quantity $E^{\kappa_1\kappa_2,l_1l_2,j}_{\sigma}$, which contains several factors of the right-hand side of (16) by

$$D^{\kappa_1\kappa_2}_{\sigma} \equiv 6\,\delta_{jj_1}\delta_{j_1j_2}\,\delta_{mm_1}\delta_{m_1m_2}\,[2j(\kappa_1)+1][2j(\kappa_2)+1]E^{\kappa_1\kappa_2,l_1l_2,j}_{\sigma}, \tag{18}$$

and the matrix element becomes

$$M = i\int_0^{\infty}\int_0^{\infty} dr_1 dr_2 \int_{-\infty}^{+\infty} dz\, r_1^2 r_2^2 \sum_{j=1}^{\infty}\sum_{m=-j}^{+j}\sum_{l_1=j-1}^{j+1}\sum_{l_2=j-1}^{j+1} 6T^{\lambda_1}_{jl_1m}\left(T^{\lambda_2}_{jl_2m}\right)^{*} j_{l_1}(\omega r_1) j_{l_2}(\omega r_2) \times \sum_{\kappa_1=-\infty}^{+\infty}\sum_{\kappa_2=-\infty}^{+\infty}[2j(\kappa_1)+1][2j(\kappa_2)+1] \times\{+[B^{\kappa_1-\kappa_2}_{12}(r_1,r_2,z) + B^{-\kappa_1\kappa_2}_{21}(r_1,r_2,z)]E^{\kappa_1\kappa_2,l_1l_2,j}_{+1} - [B^{-\kappa_1-\kappa_2}_{11}(r_1,r_2,z) + B^{\kappa_1\kappa_2}_{22}(r_1,r_2,z)]E^{\kappa_1\kappa_2,l_1l_2,j}_{-1}\}. \tag{19}$$

We utilize the relation

$$\sum_{m=-j}^{+j} T_{jl_1m}^{\lambda_1}\left(T_{jl_2m}^{\lambda_2}\right)^* = 2\pi(2j+1)d^j_{\lambda_1\lambda_2}(\vartheta)t^{\lambda_1}_{j,l_1}\left(t^{\lambda_2}_{j,l_2}\right)^* \quad , \tag{20}$$

where $d^j_{\lambda_1\lambda_2}(\vartheta)$ is the Wigner rotation matrix depending only on the scattering angle ϑ of the photon. Furthermore, we use the symmetry of the integrand under exchange of r_1 and r_2. Since only terms with even (l_1+l_2) contribute, we can take the real part of the product $t^{\lambda_1}_{j,l_1}\left(t^{\lambda_2}_{j,l_2}\right)^*$. The symmetry of the integrand under $r_1 \leftrightarrow r_2$ may be shown explicitly by taking (A7) into account, interchanging l_1 and l_2 and using

$$E_{+1}^{\kappa_1\kappa_2,l_2l_1,j} = E_{+1}^{\kappa_1\kappa_2,l_1l_2,j},$$
$$E_{-1}^{\kappa_1\kappa_2,l_2l_1,j} = E_{-1}^{-\kappa_1-\kappa_2,l_1l_2,j}. \tag{21}$$

We split M into two parts depending on σ and derive the final expression for the matrix element of Delbrück scattering:

$$M \equiv M_{+1} - M_{-1} \tag{22}$$

with

$$M_\sigma = i\int_0^\infty dr_1 \int_0^{r_1} dr_2 \int_{-\infty}^{+\infty} dz\, r_1^2 r_2^2 \sum_{j=1}^{\infty}\sum_{l_1=j-1}^{j+1}\sum_{l_2=j-1}^{j+1} 24\pi(2j+1)d^j_{\lambda_1\lambda_2}(\vartheta)j_{l_1}(\omega r_1)j_{l_2}(\omega r_2)\mathrm{Re}[t^{\lambda_1}_{j,l_1}(t^{\lambda_2}_{j,l_2})^*]$$
$$\times \sum_{\kappa_1=-\infty}^{+\infty}\sum_{\kappa_2=-\infty}^{+\infty}[2j(\kappa_1)+1][2j(\kappa_2)+1]F_\sigma^{\kappa_1\kappa_2}E_\sigma^{\kappa_1\kappa_2,l_1l_2,j} \tag{23}$$

and

$$F_{+1}^{\kappa_1\kappa_2} = B_{12}^{\kappa_1-\kappa_2}(r_1,r_2,z) + B_{21}^{-\kappa_1\kappa_2}(r_1,r_2,z),$$
$$F_{-1}^{\kappa_1\kappa_2} = B_{11}^{-\kappa_1-\kappa_2}(r_1,r_2,z) + B_{22}^{\kappa_1\kappa_2}(r_1,r_2,z). \tag{24}$$

From this expression it can be deduced that only the product of the polarization is important; thus, there are, as expected, a no-spin-flip and a spin-flip matrix element.

B. Integration contour

In the final expression for M the only part depending on the integration parameter z is $F_\sigma^{\kappa_1\kappa_2}$. We employ a Feynman contour, which passes around both the cut of $\mathcal{G}(z)$ and the shifted cut of $\mathcal{G}(z+\omega)$; see Fig. 2(a). This Feynman contour can be deformed [see Fig. 2(b)] and split into a part along the real axis ($i=1$) and one along the imaginary axis ($i=2$):

$$\int_{-\infty}^{+\infty} dz\, F_\sigma^{\kappa_1\kappa_2}(z \pm i\varepsilon) = \int_{C_F} dz\, F_\sigma^{\kappa_1\kappa_2}(z\pm i\varepsilon)$$
$$= \sum_{i=1}^{2}\int_{C_i} dz\, F_\sigma^{\kappa_1\kappa_2}(z\pm \mathrm{i}\varepsilon). \tag{25}$$

Now we can take the limit $\varepsilon \to 0$ and combine the contributions of the first line integral just below and just above the shifted cut by making use of

$$\mathcal{G}_\kappa^{ij}(r_1,r_2,z^*) = \left[\mathcal{G}_\kappa^{ij}(r_1,r_2,z)\right]^* \quad . \tag{26}$$

This yields the imaginary part of the Green's function with the shifted energy argument, while combining the integration along the imaginary axis to one integral yields the real part of the product of both Green's functions. So we can write the line integrals as ordinary integrals:

$$\int_{C_1} dz\, F_\sigma^{\kappa_1\kappa_2}(z) = 2i\int_0^{-\omega+m} dE\, f_{\sigma,1}(E),$$
$$\int_{C_2} dz\, F_\sigma^{\kappa_1\kappa_2}(z) = 2i\int_0^{\infty} du\, f_{\sigma,1}(ui), \tag{27}$$

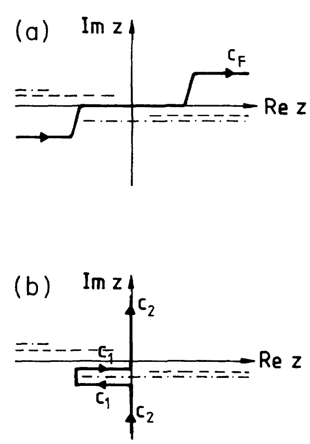

FIG. 2. Contours for the integration in the complex z plane. The dashed and the dashed-dotted lines are the cuts of the electron Green's function with energy argument z and $z+\omega$ and branch points $\pm m$ and $-\omega \pm m$, respectively. (a) shows the Feynman contour and (b) shows the deformed contour.

where m is the mass of the electron and

$$\begin{aligned} f_{+1,1}(E) &= \mathrm{Im}[\mathcal{G}^{11}_{\kappa_1}(r_2,r_1,E+\omega-i0)]\mathcal{G}^{22}_{-\kappa_2}(r_1,r_2,E-i0) \\ &\quad +\mathrm{Im}[\mathcal{G}^{22}_{-\kappa_1}(r_2,r_1,E+\omega-i0)]\mathcal{G}^{11}_{\kappa_2}(r_1,r_2,E-i0), \\ f_{-1,1}(E) &= \mathrm{Im}[\mathcal{G}^{12}_{-\kappa_1}(r_2,r_1,E+\omega-i0)]\mathcal{G}^{12}_{-\kappa_2}(r_1,r_2,E-i0) \\ &\quad +\mathrm{Im}[\mathcal{G}^{21}_{\kappa_1}(r_2,r_1,E+\omega-i0)]\mathcal{G}^{21}_{\kappa_2}(r_1,r_2,E-i0), \\ f_{+1,2}(ui) &= \mathrm{Re}[\mathcal{G}^{11}_{\kappa_1}(r_2,r_1,ui+\omega)\mathcal{G}^{22}_{-\kappa_2}(r_1,r_2,ui)] \\ &\quad +\mathrm{Re}[\mathcal{G}^{22}_{-\kappa_1}(r_2,r_1,ui+\omega)\mathcal{G}^{11}_{\kappa_2}(r_1,r_2,ui)], \\ f_{-1,2}(ui) &= \mathrm{Re}[\mathcal{G}^{12}_{-\kappa_1}(r_2,r_1,ui+\omega)\mathcal{G}^{12}_{-\kappa_2}(r_1,r_2,ui)] \\ &\quad +\mathrm{Re}[\mathcal{G}^{21}_{\kappa_1}(r_2,r_1,ui+\omega)\mathcal{G}^{21}_{\kappa_2}(r_1,r_2,ui)], \end{aligned} \tag{28}$$

denoting by $(-i0)$ the limit of the Green's function from below the cut.

III. CONCLUSIONS

Experiments on Delbrück scattering showed deviations from theoretical results in lowest-order perturbation theory and called for an improved theoretical description. In this context we have presented a general framework for the calculation of the Delbrück scattering amplitude to all orders in $(Z\alpha)$. The only approximations are the restriction to the lowest order in the fine-structure constant α and the neglect of recoil effects (external field approximation). In actual computations, however, one has to limit the number of partial waves involved. Depending on available computer facilities, our final expression may serve to determine the Delbrück scattering amplitude to high precision. Numerical calculations along this line are in progress.

ACKNOWLEDGMENTS

We thank P. J. Mohr, P. Rullhusen, and M. Schumacher for very helpful and encouraging discussions. This work was supported by Deutsche Forschungsgemeinschaft (DFG) and Bundesministerium für Forschung und Technologie (BMFT).

APPENDIX: GREEN'S FUNCTIONS

Within the Wichmann-Kroll formalism [12] one can decompose the Green's function of the electron-positron field in an external electromagnetic field into partial waves. One starts with the Fourier transform of the Green's function concerning the time variable

$$iS_F(x_2,x_1) = \frac{1}{2\pi i}\int_{C_F} dz\,\mathcal{G}(\mathbf{x}_2,\mathbf{x}_1,z)\gamma_0 e^{-iz(t_2-t_1)} \tag{A1}$$

with

$$\mathcal{G}(\mathbf{x}_2,\mathbf{x}_1,z) = \sum_{\kappa=-\infty}^{+\infty} \begin{pmatrix} \mathcal{G}^{11}_\kappa(r_2,r_1,z)\pi_{\kappa,\kappa}(2,1) & (-i)\mathcal{G}^{12}_\kappa(r_2,r_1,z)\pi_{\kappa,-\kappa}(2,1) \\ i\mathcal{G}^{21}_\kappa(r_2,r_1,z)\pi_{-\kappa,\kappa}(2,1) & \mathcal{G}^{22}_\kappa(r_2,r_1,z)\pi_{-\kappa,-\kappa}(2,1) \end{pmatrix} \tag{A2}$$

and

$$\pi_{\kappa_1,\kappa_2}(2,1) = \sum_{\mu=-(\kappa-\frac{1}{2})}^{\kappa-\frac{1}{2}} \chi_{\kappa_1\mu}(\Omega_2)\,[\chi_{\kappa_2\mu}(\Omega_1)]^\dagger, \quad |\kappa_1| = |\kappa_2| = \kappa. \tag{A3}$$

The components $\mathcal{G}^{ij}_\kappa(r_2,r_1,z)$ of the radial Green's function for $r_1 \geq r_2$ are given by

$$\begin{aligned} \mathcal{G}^{11}_\kappa(r_2,r_1,z) &= g_o(r_2)g_i(r_1)/W, \\ \mathcal{G}^{12}_\kappa(r_2,r_1,z) &= g_o(r_2)f_i(r_1)/W, \\ \mathcal{G}^{21}_\kappa(r_2,r_1,z) &= f_o(r_2)g_i(r_1)/W, \\ \mathcal{G}^{22}_\kappa(r_2,r_1,z) &= f_o(r_2)f_i(r_1)/W, \end{aligned} \tag{A4}$$

where g and f are the upper and lower components of the solution of the radial Dirac equation in the external potential $V(r)$:

$$\begin{aligned} [E-V(r)-m]\,g(r) &= \left(-\frac{d}{dr}-\frac{1}{r}+\frac{\kappa}{r}\right)f(r), \\ [E-V(r)+m]\,f(r) &= \left(+\frac{d}{dr}+\frac{1}{r}+\frac{\kappa}{r}\right)g(r). \end{aligned} \tag{A5}$$

The subscripts o and i denote regularity of the solution at the origin and at infinity, respectively. W designates the Wronskian

$$W = [f_o(r)g_i(r) - g_o(r)f_i(r)]\,r^2, \tag{A6}$$

which is independent of the radial coordinate r. The components for $r_1 < r_2$ are obtained by

$$\mathcal{G}^{ij}_\kappa(r_2,r_1,z) = \mathcal{G}^{ji}_\kappa(r_1,r_2,z) \quad . \tag{A7}$$

The Green's function for pointlike nuclei (pure Coulomb potential) were presented by Wichmann and Kroll [12] and by Mohr [13]. Finite nuclear-size effects can be obtained by using Green's functions in the potential of a spherical charge distribution. Taking a homogenously charged spherical shell as a model of the nucleus, the corresponding Green's functions were obtained by Gyulassy [17] and by Soff and Mohr [18]. Free radial Green's functions were explicitly indicated, e.g., by Mohr [13]. The latter are important in Delbrück scattering calculations, since the matrix element (1) contains an infinite part due to the free electron loop for forward scattering ($\vartheta = 0$). In numerical calculations of M for all scattering angles the same expression should be subtracted, replacing Green's functions in the external field by free radial Green's functions in order to achieve a better convergence of the sums over κ_1 and κ_2.

The radial Green's functions are analytic functions of z except for the eigenvalues of the Dirac equation. They have simple poles at the locations of the discrete eigenvalue spectrum and branch points at $z = \pm m$. There are two branch cuts in the complex plane for $z \geq m$ and $z \leq -m$, at all other points the radial Green's functions are single valued.

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